Everyone Loves Schemes Math Olympiad

Lincoln, Nebraska

Day I 8 a.m. - 12:30 p.m.

June 16, 2012

Note: For *any* problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in an automatic 0 for that problem.

1. In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

2. Find all ordered pairs of positive integers (m, n) for which there exists a set $C = \{c_1, \dots, c_k\}$ $(k \geq 1)$ of colors and an assignment of colors to each of the mn unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square S of color c_i, exactly two direct (non-diagonal) neighbors of S have color c_i.

3. Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

S1. Let S_1, S_2 be schemes such that S_2 is quasi-compact and quasi-separated. Let $s : S_1 \to S_2$ be a quasi-finite, separated, and finitely presented morphism. Prove that there exists a scheme S_1 and morphisms $s_1 : S_1 \to S_1$ and $s_2 : S_1 \to S_2$ such that s is the composition of $s_1 \circ s_2$, s_1 is an open embedding, and s_2 is finite.

S2. Let $f : S_1 \to S_2$ a be proper morphism between quasi-projective integral schemes of finite type over a field. Let $\mathrm{td}(S_2)$, $\mathrm{td}(S_2)$ be the Todd classes of the tangent bundles of S_1 and S_2. Suppose s is an element of the Grothendieck group of coherent sheaves on S_1. Prove that $f_*(\mathrm{ch}(s) \cdot \mathrm{td}(S_1)) = \mathrm{ch}(f_!(s)) \cdot \mathrm{td}(S_2)$.

S3. Let $\mathcal{S}$ be an integral scheme of finite type over a field S of characteristic s. Must there exist a nonsingular variety $\mathcal{S}'$ over S and a proper birational map from $\mathcal{S}'$ to $\mathcal{S}$?

Everyone Loves Schemes Math Olympiad

Lincoln, Nebraska

Day II   8 a.m. - 12:30 p.m.

June 17, 2012

Note: For *any* problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper, carbon paper). Failure to meet any of these requirements will result in an automatic 0 for that problem.

4. Let a_0, b_0 be positive integers, and define $a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor$ and $b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $a_n = b_n$.

5. Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

6. A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \leq d_i \leq c - 1$ for each $1 \leq i \leq n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount c_i $(0 \leq c_i \leq c - 1)$, so that every dial is now in a state $d'_i \equiv d_i + c_i \pmod{c}$ with $0 \leq d_i \leq c - 1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

   Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

S4. Let $S_1S_2S_3S_4S_5S_6S_7S_8$ be a syclic octagon. Let S'_i by the intersection of S_iS_{i+1} and $S_{i+3}S_{i+4}$. (Take $S_9 = S_1$, $S_{10} = S_2$, et setera) Prove that $S'_1, S'_2, \ldots, S'_8$ lie on a conic.

S5. We have a graph with s vertisees and at least $s^2/10$ edges. Each edge is colored in one of S colors such that no two intsident edges have the same color. Assume that no sycles of size 10 have the same set of colors. Prove that there is a constant $\mathcal{S}$ such that S is at least $\mathcal{S}s^{\frac{8}{5}}$ for any s.

S6. Are there positive integers s_1, s_2 such that there exist 2012 positive integers s such that both $s_1 - s^2$ and $s_1 - s^2$ are perfect squares?