English Language Master's Open: The Shortlist

June 18 & 19, 2011

Note: problems in each section are only very loosely ordered by difficulty.

Note: Our Engrish level beginner. Please excuse us any typos and us help fix mistake.

1 Algebra

A1 (David Yang) Let n be a positive integer. There are n soldiers stationed on the nth roots of unity in the complex plane. Each round, you pick a point, and all the soldiers shoot in a straight line towards that point; if their shot hits another soldier, the soldier hit dies and no longer shoots during the next round. What is the minimum number of rounds, in terms of n, required to eliminate all the soldiers?

A2 *ELMO 4* (Calvin Deng) Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive reals, such that whenever $a > b > c > d > 0$ are reel numbers with $ad = bc$,

$$f(a + d) + f(b - c) = f(a - d) + f(b + c).$$

A3 (Yongyi Chen) Let $p(x)$ and $q(x)$ be relatively prime polynomials polynomials such that $p(x)$ is squarefree and divides $p(q(x))$. Determine if $p(x)$ also divides at least one of $q(x) - x, q(q(x)) - x, q(q(q(x))) - x, \ldots$. (A polynomial $f(x)$ is called squarefree if there exists no nonconstant polynomial $r(x)$ such that $[r(x)]^2$ divides $f(x)$).

A4 (Evan O'Dorney) Let N be a positive integer. Define a sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0, a_1 = 1$, and $a_{n+1} + a_{n-1} = a_n(2 - \frac{1}{N})$. Prove that $a_n < \sqrt{N+1}$ for all n.

A5 (Victor Wang) Given positive reals x, y, z such that $xy + yz + zx = 1$, show that

$$\sum_{\text{cyc}} \sqrt{(xy + kx + ky)(xz + kx + kz)} \geq k^2,$$

where $k = 2 + \sqrt{3}$.

A6 (Calvin Deng) Let $Q(x)$ be a polynomials with integer coefficients. Prove that there exists a polynomial $P(x)$ also with integer coefficients such that when $n \geq \deg(Q(x))$,

$$\sum_{i=0}^{n} \frac{!iP(i)}{i!(n-1)!} Q(n),$$

where $!i$ notes the number of derangements (permutations with no fixed points) of $1, 2, \ldots, n$.

A7 *ELMO 3* (Alex Zhu) Determine whether there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers such that the following holds:

- For all $n \geq 0$, $a_n \neq 0$.
- There exist real numbers x and y such that $a_{n+2} = xa_{n+1} + ya_n$ for all $n \geq 0$.
- For all positive real numbers r, there exists positive integers i and j such that $|a_i| < r < |a_j|$.

A7 (Evan O'Dorney) Let $n > 1$ be an integer. Three complex numbers have the property that both their sum and the sum of their n^{th} powers are 0. Prove that two of the three numbers have the same magnitude.

2 Combinatorics

C1 (Evan O'Dorney) Let S be a finite set, and let F be a family of subsets of S such that:

- If $A \subseteq S$, then $A \in F$ if and only if $S \backslash A \notin F$.
- If $A \subseteq B \subseteq S$ and $B \in F$, then $A \in F$.

Determine if there must exist a function $f : S \to \mathbb{R}$ such that, for $A \subseteq S$,

$$A \in F \text{if and only if} \sum_{s\in A} f(s) < \sum_{s\in S\backslash A} f(s).$$

C2 (David Yang) A directed graph has each vertex with outdegree 2. Prove that it is possible to split the vertices into 3 sets so that for each vertex V, V is not simultaneously in the same set with both of the vertices that it points to.

C3 *ELMO 2* (Linus Hamilton) Wanda the Worm likes to eat Pascal's triangle. One day, she starts at the top of the triangle and eats $\binom{0}{0} = 1$. Each move, she travels to an adjacent positive integer and eats it, but she can never return to a spot that she has previously eaten. If Wanda can never eat numbers a, b, c such that $a + b = c$, proof that it is possible for her to eat $100,000$ numbers in the first 2011 rows given that she is not restricted to traveling only in the first 2011 rows.

C4 (David Yang) Prove that we can split any graph with $n > 2$ vertices into a forest and k disjoint cycles, where $k < cn\ln(n)$ for some constant c independant of n. (Harder version: $k < cn$)

C5 (David Yang) Do there exist positive integers k and n such that for any finite graph with diameter $k+1$ it is possible to choose n vertices such that from any unchosen vertex there is a path to a chosen vertex with length at most k?

C6 (David Yang) Let T be a tree. Prove that there is a constant such that every graph with n vertices that does not contain a subgraph isomorphic to T has at most cn edges, for some constant c independant of n.

C7 *ELMO 6* (David Yang) Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternature turns in this fashion, with Little D's goal to loose a shoe on each of n consecutive lattice points on a line parrallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

3 Geometry

G1 *ELMO 1* (Evan O'Dorney) Let $ABCD$ be a convex quadralateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be intersection of EG and FH. Given that quadrilaterals $HAEP, EBFP, FCGP, GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.

G2 (David Yang) Let $\omega, \omega_1, \omega_2$ be three mutally tangent circles such that ω_1 and ω_2 are externally tangent at P, ω_1 and ω are internally tangent at A, and ω and ω_2 are internally tangent at B. Let O, O_1, O_2 be the centers of $\omega, \omega_1, \omega_2$, respectively. Given that X is the foot of the perpendicular from P to AB, prove that $\angle O_1XP = \angle O_2XP$.

G3 (Tom Lu) Let ABC be a triangle. Let ω_A be a circle tangent to AB and AC, ω_B be a circle tangent to AB and BC, and ω_C be a circle tangent to AC and BC. Let P_A be the center of inverse homothety between ω_B and ω_C. Define P_B and P_C similarly. Prove that AP_A, BP_B, and CP_C are concurrent. (Necessary or unnecessary condition–the three circles are located inside the triangle?)

G4 (Calvin Deng) Prove that any convex pentagon has exactly one pair of isogonal conjugates.

4 Number Theory

N1 (Victor Wang) Let $p \geq$ be a prime. Show that

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{p}{k} 3^k \equiv 2^p - 1(\mod p^2).$$

N2 (Victor Wang) If x is an integer, determine when each of the following is a perfect square:

(a) $8x^4 - 8x^2 + 1$

(b) $32x^4 - 8x^2 + 1$

(c) $20x^4 - 4x^2 + 1$

N3 *ELMO 5* (Alex Zhu) Let $p > 13$ be a prime of the the form $2q + 1$, where q is prime. Find the number of ordered pairs of integers (m, n) such that $0 \leq m < n < p - 1$ and

$$3^m + (-12)^m \equiv 3^n + (-12)^n \pmod{p}.$$

N4 (Carl Lian) Determine all positive integers n such that there exists an n-digit base 10 integer $a_1 a_2 \cdots a_n$, where a_1 may be equal to 0, such that the n integers which result from cycling the digits give n distinct remainders when divdied by n.

N5 (Mitchell Lee) Given an integer $n > 1$, and n-tuple $(a_1, a_2, \ldots, a_n)$ of integers greater than 1 is called *good* if $a_i | \frac{a_i a_2 \ldots a_n}{a_i} - i$ for $i = 1, 2, \ldots, n$. Prove that there are finitely many good n-tuples for any fixed n.