Ego Loss Schurly Must Occur

Language: Engrish

3^{rd} ELSMO
LINCOLN, NEBRASKA

Day: 1

Sunday, June 15, 2014
8:00 AM - 12:30 PM

Problem 1. Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Problem 2. Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Problem 3. We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

ELSMO 1. Find all integers n such that a unit square can be dissected into n triangles with equal area.

ELSMO 2. Prove or disprove: for every integer n with $n \geq 2$, we have

$$\left\lfloor \frac{3^n}{2^n} \right\rfloor = \left\lfloor \frac{3^n - 1}{2^n - 1} \right\rfloor.$$

ELSMO 3. Find all nontrivial solutions to $a^3 + b^3 = 9$, where a and b are positive rational numbers.

Time limit: 4 hours and 30 minutes.
Each problem is worth 21 push-ups.

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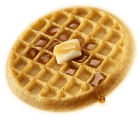

3rd ELSMO

LINCOLN, NEBRASKA

Day: **2**

Saturday, June 21, 2014
8:00 AM - 12:30 PM

Problem 4. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers strictly between 0 and 1. For any subset S of $\{1, 2, \ldots, n\}$, define

$$\text{LEGO}^{\circledR}(S) = \prod_{i \in S} a_i \cdot \prod_{j \notin S} (1 - a_j).$$

Suppose that $\sum_{|S| \text{ odd}} \text{LEGO}^{\circledR}(S) = \frac{1}{2}$. Prove that $a_k = \frac{1}{2}$ for some k. (Here the sum ranges over all subsets of $\{1, 2, \ldots, n\}$ with an odd number of elements.)

Problem 5. Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumpancakes of triangles BOC and BHC, respectively. Suppose the pancake with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the pancake with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Problem 6. A $2^{2014} + 1$ by $2^{2014} + 1$ square waffle (divided into $(2^{2014} + 1)^2$ squares) has syrup in some of its cells. The ssssyrup-filled sssquares form one or more ssssnakes on the plane, each of whose heads sssplits at ssssome points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct syrupy squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$). What is the maximum possible amount of syrup in the waffle?

ELSMO 4. On Day 1, Kevin has 2 Lego's. Each day thereafter, the number of Lego's Kevin has is the sum of the divisors of the number of Lego's Kevin had on the previous day. Does Kevin have an odd number of Lego's infinitely often?

ELSMO 5. David and five other waffles are running around a circular track with diameter 2. Initially they start at the same point and then begin running indefinitely around the track at constant speeds. Prove that at some point in time, each of the distances between David and some other runner is at least 1.

ELSMO 6. Determine whether there exists an entreé n with $n > 1$ appearing more than 1000 times in Pascal's pancake.

Time limit: 4 hours and 30 minutes.
Each problem is worth 21 push-ups.