Every Little Mistake $\implies$ 0 Shortlist

MOP 2012

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Note: The problem czars' recommendations are bolded.

1 Geometry

G1. (Ray Li) In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

G2. (Ray Li) In triangle ABC, P is a point on altitude AD. Q, R are the feet of the perpendiculars from P to AB, AC, and QP, RP meet BC at S and T respectively. the circumcircles of BQS and CRT meet QR at X, Y.

a) Prove SX, TY, AD are concurrent at a point Z.

b) Prove Z is on QR iff $Z = H$, where H is the orthocenter of ABC.

G3. (Alex Zhu) ABC is a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD = \angle DPC$.

G4. (Ray Li) Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of AB. Another circle, ω_1 is tangent to Ω, ω, and AB at D, Z, and F respectively. Rays CD and AB meet at P. If M is the midpoint of major arc AB, show that $\tan \angle ZEP = \frac{PE}{CM}$.

G5. (Calvin Deng) Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

G6. (Ray Li) In $\triangle ABC$, H is the orthocenter, and AD, BE are arbitrary cevians. Let ω_1, ω_2 denote the circles with diameters AD and BE, respectively. HD, HE meet ω_1, ω_2 again at F, G. DE meets ω_1, ω_2 again at P_1, P_2 respectively. FG meets ω_1, ω_2 again Q_1, Q_2 respectively. P_1H, Q_1H meet ω_1 at R_1, S_1 respectively. P_2H, Q_2H meet ω_2 at R_2, S_2 respectively. Let $P_1Q_1 \cap P_2Q_2 = X$, and $R_1S_1 \cap R_2S_2 = Y$. Prove that X, Y, H are collinear.

G7. (Alex Zhu) Let $\triangle ABC$ be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90°$.

2 Algebra

A1. (Ray Li, Max Schindler) Let $x_1, x_2, x_3, y_1, y_2, y_3$ be nonzero real numbers satisfying $x_1 + x_2 + x_3 = 0, y_1 + y_2 + y_3 = 0$. Prove that

$$\frac{x_1x_2 + y_1y_2}{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}} + \frac{x_2x_3 + y_2y_3}{\sqrt{(x_2^2 + y_2^2)(x_3^2 + y_3^2)}} + \frac{x_3x_1 + y_3y_1}{\sqrt{(x_3^2 + y_3^2)(x_1^2 + y_1^2)}} \geq -\frac{3}{2}$$

A2. (Owen Goff) Let a, b, c be three positive real numbers such that $a \leq b \leq c$ and $a + b + c = 1$. Prove that

$$\frac{a + c}{\sqrt{a^2 + c^2}} + \frac{b + c}{\sqrt{b^2 + c^2}} + \frac{a + b}{\sqrt{a^2 + b^2}} \leq \frac{3\sqrt{6}(b + c)^2}{\sqrt{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}}.$$

A3. (David Yang) Let a_0, b_0 be positive integers, and define $a_{i+1} = a_i + \lfloor\sqrt{b_i}\rfloor$ and $b_{i+1} = b_i + \lfloor\sqrt{a_i}\rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $a_n = b_n$.

A4. (David Yang) Prove that if m, n are relatively prime positive integers, $x^m - y^n$ is irreducible in the complex numbers. (A polynomial $P(x, y)$ is irreducible if there do not exist nonconstant polynomials $f(x, y)$ and $g(x, y)$ such that $P(x, y) = f(x, y)g(x, y)$ for all x, y.)

A5. (Calvin Deng) Let $a, b, c \geq 0$. Show that

$$(a^2 + 2bc)^{2012} + (b^2 + 2ca)^{2012} + (c^2 + 2ab)^{2012} \leq (a^2 + b^2 + c^2)^{2012} + 2(ab + bc + ca)^{2012}.$$

A6. (Victor Wang) Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

Note: A4 is a special case of A6, but is significantly easier.

A7. (Alex Zhu) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x)f(y)f(x + y) = f(xy)(f(x) + f(y))$ for all $x, y \in \mathbb{Q}$.

A8. (David Yang) Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a cyclic octagon. Let B_i by the intersection of A_iA_{i+1} and $A_{i+3}A_{i+4}$. (Take $A_9 = A_1$, $A_{10} = A_2$, etc.) Prove that $B_1, B_2, \ldots, B_8$ lie on a conic.

3 Number Theory

N1. (David Yang, Alex Zhu) Find all positive integers n such that $4^n + 6^n + 9^n$ is a square.

N2. (Anderson Wang) For positive rational x, if x is written in the form $\frac{p}{q}$ with p, q positive relatively prime integers, define $f(x) = p + q$. For example, $f(1) = 2$. Prove that if $f(x) = f(\frac{mx}{n})$ for rational x and positive integers m, n, then $f(x)$ divides $|m - n|$.

Possible part (b): Let n be a positive integer. If all x which satisfy $f(x) = f(2^n x)$ also satisfy $f(x) = 2^n - 1$, find all possible values of n.

N3. (Alex Zhu) Let $s(k)$ be the number of ways to express k as the sum of distinct 2012^{th} powers. Show that for every real number c there exists an integer n such that $s(n) > cn$.

N4. (Lewis Chen) Do there exist positive integers $b, n > 1$ such that when n is expressed in base b, there are more than n distinct permutations of its digits? For example, when $b = 4$ and $n = 18$, $18 = 102_4$, but 102 only has 6 digit arrangements. (Leading zeros are allowed in the permutations.)

N5. (Ravi Jagadeesan) Let $n > 2$ be a positive integer and let p be a prime. Suppose that the nonzero integers are colored in n colors. Let $a_1, a_2, \ldots, a_n$ be integers such that for all $1 \le i \le n$, $p^i \nmid a_i$ and $p^{i-1} \mid a_i$. In terms of n, p, and $\{a_i\}_{i=1}^{n}$, determine if there must exist integers $x_1, x_2, \ldots, x_n$ of the same color such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

N6. (Calvin Deng) Prove that if a and b are positive integers and $ab > 1$, then

$$\left\lfloor \frac{(a-b)^2 - 1}{ab} \right\rfloor = \left\lfloor \frac{(a-b)^2 - 1}{ab - 1} \right\rfloor$$

Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

N7. (Bobby Shen) A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \le d_i \le c-1$ for each $1 \le i \le n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer amount c_i $(0 \le c_i \le c - 1)$, so that every dial is now in a state $d'_i \equiv d_i + c_i \pmod{c}$ with $0 \le d'_i \le c - 1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

N8. (Victor Wang) Fix two positive integers $a, k \ge 2$, and let $f \in \mathbb{Z}[x]$ be a polynomial. Suppose that for all sufficiently large positive integers n, there exists a rational number x satisfying $f(x) = f(a^n)^k$. Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $f(g(x)) = f(x)^k$ for all real x.

N9. (David Yang) Are there positive integers m, n such that there exist 2012 positive integers x such that both $m - x^2$ and $n - x^2$ are perfect squares?

4 Combinatorics

C1. (David Yang) Let $n \geq 2$ be a positive integer. Given a sequence s_i of n distinct real numbers, define the "class" of the sequence to be the sequence $a_1, a_2, \ldots, a_{n-1}$, where a_i is 1 if $s_{i+1} > s_i$ and -1 otherwise.

Find the smallest integer m such that there exists a sequence w_i such that for every possible class of a sequence of length n, there is a subsequence of w_i that has that class.

C2. (David Yang) Let A be the set of positive integers with at most 10 digits and with all digits 0 or 1. Let B be the set of positive integers with at most 10 digits and with all digits 0,1,2, or 3. Define the difference set $X - Y$ of two sets of reals X, Y to be the set of elements z of the form $x - y$, where $x \in X$ and $y \in Y$. Prove that for any finite set of positive integers C, $|C - A| \leq |C - B| \leq 1024|C - A|$.

C3. (David Yang) Find all ordered pairs of positive integers (m, n) for which there exists a set $C = \{c_1, \ldots, c_k\}$ $(k \geq 1)$ of colors and an assignment of colors to each of the mn unit squares of a $m \times n$ grid such that for every color $c_i \in C$ and unit square S of color c_i, exactly two direct (non-diagonal) neighbors of S have color c_i.

C4. (Calvin Deng) A tournament on $2k$ vertices contains no 7-cycles. Show that its vertices can be partitioned into two sets, each with size k, such that the edges between vertices of the same set do not determine any 3-cycles.

C5. (Linus Hamilton) Form the infinite graph A by taking the set of primes p congruent to $1 \pmod 4$, and connecting p and q if they are quadratic residues modulo each other. Do the same for a graph B with the primes $1 \pmod 8$. Show A and B are isomorphic to each other.

C6. (Linus Hamilton) Consider a directed graph G with n vertices, where 1-cycles and 2-cycles are permitted. For any set S of vertices, let $N^+(S)$ denote the out-neighborhood of S (i.e. set of successors of S), and define $(N^+)^k(S) = N^+((N^+)^{k-1}(S))$ for $k \geq 2$.

For fixed n, let $f(n)$ denote the maximum possible number of distinct sets of vertices in $\{(N^+)^k(X)\}_{k=1}^{\infty}$. Show that there exists $n > 2012$ such that $f(n) < 1.0001^n$.

C7. (David Yang) We have a graph with n vertices and at least $n^2/10$ edges. Each edge is colored in one of c colors such that no two incident edges have the same color. Assume that no cycles of size 10 have the same set of colors. Prove that there is a constant k such that c is at least $kn^{\frac{8}{5}}$ for any n.

C8. (Victor Wang) Consider the equilateral triangular lattice in the complex plane defined by the Eisenstein integers; let the ordered pair (x, y) denote the complex number $x + y\omega$ for $\omega = e^{2\pi i/3}$. We define an ω-chessboard polygon to be a (non self-intersecting) polygon whose sides are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit triangles, which are shaded alternately black and white so that adjacent triangles have different colors. To tile an ω-chessboard polygon by lozenges is to exactly cover the polygon by non-overlapping rhombuses consisting of two bordering triangles. Finally, a *tasteful tiling* is one such that for every unit hexagon tiled by three lozenges, each lozenge has a black triangle on its left (defined by clockwise orientation) and a white triangle on its right (so the lozenges are BW, BW, BW in clockwise order).

a) Prove that if an ω-chessboard polygon can be tiled by lozenges, then it can be done so tastefully.

b) Prove that such a tasteful tiling is unique.