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The second term in asymptotic expansion
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We will denote A.E. for asymptotic expansion

Application 1:

$$x_0 = 1 \ , \ x_{n+1} = x_n + \frac{1}{x_n}$$

All the terms are strictly positives , the sequence (x_n) is increasing
Suppose it bounded then it will converge to $L > 0$, with $L = L + \frac{1}{L}$
contradiction.

$$\lim_{n \to +\infty} x_n = +\infty$$

$$x_{n+1}^2 - x_n^2 = 2 + \frac{1}{x_n^2} \to 2 \text{ when } n \to +\infty$$

With Cesaro theorem

$$\frac{1}{n}\sum_{k=0}^{n-1}(x_{k+1}^2 - x_k^2) \to 2 \text{ when } n \to +\infty$$

The equivalent is

$$x_n \sim \sqrt{2n}$$

$$x_{n+1}^2 - x_n^2 - 2 = \frac{1}{x_n^2} \sim \frac{1}{2n}$$

Now we use equivalent of partial sum for divergent series with constant sign.

$$\sum_{k=1}^{n-1}(x_{k+1}^2 - x_k^2 - 2) \sim \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k} \sim \frac{1}{2}\ln(n)$$

$$x_n^2 - 2n \sim \frac{1}{2}\ln(n)$$

$$x_n^2 = 2n + \frac{1}{2}\ln(n) + o(\ln(n))$$

$$x_n = \sqrt{2n + \frac{1}{2}\ln(n) + o(\ln(n))} = \sqrt{2n}\left(1 + \frac{1}{2n}\ln(n) + o\left(\frac{\ln(n)}{n}\right)\right)^{\frac{1}{2}}$$

$$x_n = \sqrt{2n}\left(1 + \frac{1}{2}\frac{1}{2n}\ln(n) + o\left(\frac{\ln(n)}{n}\right)\right)$$

$$x_n = \sqrt{2n} + \frac{\sqrt{2}}{4\sqrt{n}}\ln(n) + o\left(\frac{\ln(n)}{\sqrt{n}}\right)$$

The second term in asymptotic expansion is $\frac{\sqrt{2}}{4\sqrt{n}}\ln(n)$

Application 2 Let $x_0 = 1$, $x_{n+1} = \sin(x_n)$

All terms of (x_n) are strictly positive , decreasing then the limit of (x_n) exist $L \geq 0$ and $L = \sin(L)$ so the limit is $L = 0$

Let's find a real a such that

$$x_{n+1}^a - x_n^a \to k \neq 0 \text{ when } n \to +\infty$$

$$(1)\quad x_{n+1}^a - x_n^a = (\sin(x_n))^a - x_n^a$$

$$x_{n+1}^a - x_n^a = \left(x_n - \frac{x_n^3}{6} + o(x_n^3)\right)^a - x_n^a$$

$$x_{n+1}^a - x_n^a = x_n^a\left(\left(1 - \frac{x_n^2}{6} + o(x_n^2)\right)^a - 1\right)$$

$$x_{n+1}^a - x_n^a = x_n^a\left(1 - a\frac{x_n^2}{6} + o(x_n^2) - 1\right)$$

$$x_{n+1}^a - x_n^a = -a\frac{x_n^{2+a}}{6} + o(x_n^{2+a})$$

Let's take $a = -2$ then

$$\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2} = \frac{1}{3} + o(1) \to \frac{1}{3} \text{ when } n \to +\infty$$

With the Cesaro theorem

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{1}{x_{k+1}^2}-\frac{1}{x_k^2}\right)\to\frac{1}{3}\text{ when } n\to+\infty$$

(2) $x_n \sim \sqrt{\frac{3}{n}}$ or $x_n = \sqrt{\frac{3}{n}}+o\left(\frac{1}{n}\right)$

Now we take one more term in asymptotic expansion in (1)

$$x_{n+1}^a - x_n^a = \left(x_n - \frac{x_n^3}{6}+\frac{x_n^5}{120}+o(x_n^5)\right)^a - x_n^a$$

$$x_{n+1}^a - x_n^a = x_n^a\left(\left(1-\frac{x_n^2}{6}+\frac{x_n^4}{120}+o(x_n^4)\right)^a-1\right)$$

$$x_{n+1}^a - x_n^a = x_n^a\left(1-a\frac{x_n^2}{6}+a\frac{x_n^4}{120}+\frac{a(a-1)}{2}\left(\frac{x_n^2}{6}\right)^2+o(x_n^4)-1\right)$$

$$x_{n+1}^a - x_n^a = -a\frac{x_n^{2+a}}{6}+a\frac{x_n^{4+a}}{120}+\frac{a(a-1)}{2}\frac{x_n^{4+a}}{36}+o(x_n^4)$$

with $a=-2$

$$\frac{1}{x_{n+1}^2}-\frac{1}{x_n^2}=\frac{1}{3}+\frac{x_n^2}{15}+o(x_n^2)$$

with (2)

$$\frac{1}{x_{n+1}^2}-\frac{1}{x_n^2}-\frac{1}{3}\sim\frac{1}{15}\frac{3}{n}$$

$$\frac{1}{x_{n+1}^2}-\frac{1}{x_n^2}-\frac{1}{3}\sim\frac{1}{5n}$$

Now we use equivalent of partial sum for divergent series with constant sign.

$$\sum_{k=1}^{n-1}\left(\frac{1}{x_{k+1}^2}-\frac{1}{x_k^2}-\frac{1}{3}\right)\sim\frac{1}{5}\sum_{k=1}^{n-1}\frac{1}{k}\sim\frac{\ln(n)}{5}$$

$$\frac{1}{x_n^2}-\frac{n}{3}=\frac{\ln(n)}{5}+o(\ln(n))$$

$$x_n=\frac{1}{\sqrt{\frac{n}{3}+\frac{\ln(n)}{5}+o(\ln(n))}}$$

$$x_n=\sqrt{\frac{3}{n}}\left(1+\frac{3\ln(n)}{5n}+o\left(\frac{\ln(n)}{n}\right)\right)^{-\frac{1}{2}}$$

$$x_n=\sqrt{\frac{3}{n}}\left(1-\frac{1}{2}\frac{3\ln(n)}{5n}+o\left(\frac{\ln(n)}{n}\right)\right)$$

$$x_n=\sqrt{\frac{3}{n}}-\frac{3\sqrt{3}}{10}\frac{\ln(n)}{n^{\frac{3}{2}}}+o\left(\frac{\ln(n)}{n^{\frac{3}{2}}}\right)$$

the second term in A.E. is $-\frac{3\sqrt{3}}{10}\frac{\ln(n)}{n^{\frac{3}{2}}}$

Application 3 $x_0 \in \mathbb{R}+$, $x_{n+1} = x_n + e^{-x_n}$

All terms of (x_n) are strictly positive , increasing if (x_n) is bounded then it will converge to $L = L + e^{-L}$ contradiction.

$$\lim x_n = +\infty$$

Here it is not possible to find a real a such that

$$x_{n+1}^a - x_n^a \to k \neq 0 \text{ when } n \to +\infty$$

You should ask why ? I will explain to you

$$\begin{aligned}
x_{n+1}^a - x_n^a &= (x_n + e^{-x_n})^a - x_n^a \\
x_{n+1}^a - x_n^a &= x_n^a\left(\left(1 + \frac{e^{-x_n}}{x_n}\right)^a - 1\right) \\
x_{n+1}^a - x_n^a &= x_n^a\left(1 + a\frac{e^{-x_n}}{x_n} + o\left(\frac{e^{-x_n}}{x_n}\right) - 1\right) \\
(1)\quad x_{n+1}^a - x_n^a &= a x_n^{a-1} e^{-x_n} + o(x_n^{a-1} e^{-x_n})
\end{aligned}$$

For any real a

$$\lim_{n\to+\infty} x_n^{a-1} e^{-x_n} = 0 \text{ since } \lim_{n\to+\infty} x_n = +\infty$$

We have to find a function F such that

$$F(x_{n+1}) - F(x_n) \to k \neq 0 \text{ when } n \to +\infty$$

with F invertible around $+\infty$

At this stage I can give you the function F , but I prefer to explain the general idea to find F .

Consider the differential equation associate to $x_{n+1} = x_n + e^{-x_n}$

that is $y' = e^{-y}$ this gives $y'e^{y} = 1$ integrate it around $+\infty$ you get

$e^{y(x)} \sim x$ now we have the function $F(t) = e^{t}$ wich will gives us the key

$$\begin{aligned}
F(x_{n+1}) - F(x_n) &= e^{x_{n+1}} - e^{x_n} \\
e^{x_{n+1}} - e^{x_n} &= e^{x_n + e^{-x_n}} - e^{x_n} \\
(1) \qquad e^{x_{n+1}} - e^{x_n} &= e^{x_n}(e^{e^{-x_n}} - 1) \\
e^{x_{n+1}} - e^{x_n} &= e^{x_n}(1 + e^{-x_n} + o(e^{-x_n}) - 1) \\
e^{x_{n+1}} - e^{x_n} &= 1 + o(1) \to 1 \quad \text{when } n \to +\infty
\end{aligned}$$

With Cesaro theorem

$$\frac{1}{n}\sum_{k=0}^{n-1}(e^{x_{k+1}} - e^{x_k}) \to 1 \quad \text{when } n \to +\infty$$

$$e^{x_n} \sim n \quad \text{then } x_n \sim \ln(n)$$

Taking one more term in (1)

$$\begin{aligned}
e^{x_{n+1}} - e^{x_n} &= e^{x_n}\left(1 + e^{-x_n} + \frac{1}{2}e^{-2x_n} + o(e^{-2x_n}) - 1\right) \\
e^{x_{n+1}} - e^{x_n} &= 1 + \frac{1}{2}e^{-x_n} + o(e^{-x_n}) \\
e^{x_{n+1}} - e^{x_n} - 1 &\sim \frac{1}{2n}
\end{aligned}$$

Again with we use equivalent of partial sum for divergent series with constant sign.

$$\begin{aligned}
\sum_{k=1}^{n}(e^{x_{k+1}} - e^{x_k} - 1) &\sim \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} \sim \frac{1}{2}\ln(n) \\
e^{x_{n+1}} - e^{x_1} - n &\sim \frac{1}{2}\ln(n) \\
e^{x_{n+1}} &= n + \frac{1}{2}\ln(n) + o(\ln(n)) \\
x_n &= \ln\left(n + \frac{1}{2}\ln(n) + o(\ln(n))\right) \\
x_n &= \ln(n) + \ln\left(1 + \frac{\ln(n)}{2n} + o\left(\frac{\ln(n)}{n}\right)\right) \\
x_n &= \ln(n) + \frac{\ln(n)}{2n} + o\left(\frac{\ln(n)}{n}\right)
\end{aligned}$$

The second term in A.E is $\frac{\ln(n)}{2n}$ beautiful .

This exercice was given at Oral Examination Ecole Polytechnique, France.

Exercice 1: Try to find the third term in the three applications above.

Exercice 2: $k \geq 1$ integer , $x_0 > 0$, $x_{n+1} = x_n + \frac{1}{\sqrt[k]{x_n}}$

Find an equivalent of (x_n) and the second , the third term in asymptotic expansion of (x_n).

This sequence was use in Putnam 2006 B6 (U.S.A) it was asked find

$$\lim_{n \to +\infty} \frac{x_n^{k+1}}{n^k}$$

Exercice 3: $x_0 = 2$, $x_{n+1} = x_n + \ln(x_n)$ Find an equivalent of (x_n) and the second , third term in asymptotic expansion.

This one was Oral Examination 2010 Ecole Centrale , Paris France.

Exercice4: $a_0(x) = x \in [0;\pi]$, $a_{n+1}(x) = \sin(a_n(x))$. Prove that there exist a function $\varphi(x)$ such that we have the asymptotic expansion uniformly in $x \in [\alpha;\pi - \alpha]$ where $0 < \alpha < \frac{\pi}{2}$

$$\frac{1}{a_n^2(x)} = \frac{n}{3} + \frac{\ln(n)}{5} + \varphi(x) + \frac{3}{25}\frac{\ln(n)}{n} + o\left(\frac{\ln(n)}{n}\right)$$

Study the continuity of φ on $]0;\pi[$. Study the monotony , the convexity of φ on $]0;\frac{\pi}{2}]$. Study the function $\varphi(\sin x) - \varphi(x)$.

This one was from Training for Agrégation of Mathematics.